2.4 Directional derivatives and Gradient

Tuesday, February 16, 2021 12:11 PM

Definition. For a vector $\vec{v}$ and a function f defined on a domain D containing (a,b)

We define $D_{\vec{v}} f(a,b) =$

$$\lim_{h \to 0} \frac{f((a,b) + h\vec{v}) - f(a,b)}{h}$$

$$= \lim_{h \to 0} \frac{f(a + hv_1, b + hv_2) - f(a,b)}{h}$$

"rate of change of f in the direction $\vec{v}$"

Notice

$$\frac{\partial f}{\partial x} = D_{\hat{i}} f \qquad \frac{\partial f}{\partial y} = D_{\hat{j}} f \qquad \text{in 3D } \frac{\partial f}{\partial z} = D_{\vec{k}} f$$

What about $D_{\vec{v}} f$ for:

- $\vec{v} = 2\hat{i}$

$$\lim_{h \to 0} \frac{f(a + 2h, b) - f(a,b)}{h}$$

$h' = 2h$

$$\hookrightarrow \lim_{h' \to 0} \frac{f(a + h', b) - f(a,b)}{h'/2}$$

$$= \lim_{h' \to 0} 2 \cdot \frac{f(a + h', b) - f(a,b)}{h'}$$

$$= 2 \lim_{h' \to 0} \ldots = 2 \frac{\partial f}{\partial x}$$

- $\vec{v} = c\hat{i}$, $c \in \mathbb{R}$

then $D_{\vec{v}} f = c \dfrac{\partial f}{\partial x}$

$$D_{c\vec{v}} f = c D_{\vec{v}} f$$

- Notice $D_{2\vec{v}} = 2 D_{\vec{v}} f$ can be written as

$$D_{\vec{v} + \vec{v}} = D_{\vec{v}} f + D_{\vec{v}} f$$

Conjecture

$$D_{\vec{v} + \vec{w}} f = D_{\vec{v}} f + D_{\vec{w}} f$$

Equivalently

Say $\vec{v} = v_1 \vec{i} + v_2 \vec{j}$

$$D_{\vec{v}} f = D_{v_1 \vec{i}} f + D_{v_2 \vec{j}} f$$

$$= v_1 D_{\vec{i}} f + v_2 D_{\vec{j}} f$$

$$= v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y}$$

$$= \vec{v} \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

In fact, if $D_{\vec{v}} f = v_1 \partial f \ldots , v_2 \partial f$

$$= \vec{v}\cdot\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

In fact, if $D_{\vec{v}} f = v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y}$ and $D_{\vec{w}} f = w_1 \frac{\partial f}{\partial x} + w_2 \frac{\partial f}{\partial y}$ then the conjecture $D_{\vec{v}+\vec{w}} f = D_{\vec{v}} f + D_{\vec{w}} f$ is true

Let's prove $D_{\vec{v}} f = v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y}$

$$D_{\vec{v}} f = \lim_{h\to 0} \frac{f(a+hv_1, b+hv_2) - f(a,b)}{h}$$

Idea: going from $(a,b) \to (a+hv_1, b+hv_2)$ can be done in steps $(a,b) \to (a+hv_1, b)$
$\to (a+hv_1, b+hv_2)$

$$= \lim_{h\to 0} \frac{[f(a+hv_1, b+hv_2) - f(a+hv_1, b)] + [f(a+hv_1, b) - f(a,b)]}{h}$$

$$= \lim_{h\to 0} \frac{f(a+hv_1, b+hv_2) - f(a+hv_1, b)}{h} + \lim_{h\to 0} \frac{f(a+hv_1, b) - f(a,b)}{h}$$

Let's do each separately:

$$\lim_{h\to 0} \frac{f(a+hv_1, b) - f(a,b)}{h} = D_{(v_1, 0)}$$

$$= D_{v_1 \vec{i}} f = v_1 D_{\vec{i}} f = v_1 \frac{\partial f}{\partial x}$$

$$\lim_{h\to 0} \frac{f(a+hv_1, b+hv_2) - f(a+hv_1, b)}{h}$$

glossing over some technicalities

$$= \lim_{h\to 0} D_{(0,v_2)} f(x+hv_1, y)$$

$$= \lim_{h\to 0} v_2 \frac{\partial f}{\partial y}(a+hv_1, b)$$

$$= v_2 \frac{\partial f}{\partial y}(a,b)$$

if $\frac{\partial f}{\partial y}$ is continuous

Conclusion If f is differentiable at and near (a,b) and the partial derivative are continuous near (a,b) then

$$D_{\vec{v}} f(a,b) = v_1 \frac{\partial f}{\partial x}(a,b) + v_2 \frac{\partial f}{\partial y}(a,b)$$

$\Rightarrow$ Conjecture is true

Conclusion: $D_{\vec{v}} f = \vec{v} \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is called gradient of f and denoted ∇f

$\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}$

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is called gradient of f and denoted ∇f

In 3D: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

In general: $D_{\vec{v}} f = \vec{v} \cdot \nabla f$ as long as partial derivatives of f are continuous (near the point in question)

Formal def of near:
we say that P happens/is true "near (a, b)" if $\exists\, \varepsilon > 0$ such that P is true for all (x, y) such that $\|(x, y) - (a, b)\| < \varepsilon$
↳ same idea in $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^n$ etc
aka – "happens in a neighborhood of"

Terminology f is "continuously differentiable in a region D" → aka "smooth"
if the partial derivatives of f exist and are continuous in D.

Geometric Meaning of Gradient

Say we want to compare $D_{\vec{v}} f$ for different $\vec{v}$

Note: $D_{\vec{v}} f = \vec{v} \cdot \nabla f$

Let's fix $\|\vec{v}\|$ and vary the direction

- $D_{\vec{v}} f$ is 0 when $\vec{v} \perp \nabla f$
- $D_{\vec{v}} f$ is biggest when $\vec{v}$ is in same direction as ∇f
- $D_{\vec{v}} f$ is smallest when $\vec{v}$ is in opposite dir as ∇f

ex If f describes temp and you're cold, then you want to go in direction of ∇f

↳ hot? → go in dir of $-\nabla f$

- level curves are always $\perp \nabla f$
  e.g. topographical map
  ∇ elevation $\perp$ lines